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Normal forms for locally exact Poisson structures in $\mathbf{R}^{3☆}$

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Abstract

We consider smooth singular Poisson tensors in $\mathbf{R}^3$ which admit a trivial curl vector field. We use normal forms for smooth functions in $\mathbf{R}^3$ to produce normal forms for such Poisson tensors around the singular point. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

A smooth Poisson structure $\{\ ,\ \}$ on a manifold M is a Lie algebra structure on $C^\infty(M)$ satisfying the Leibniz identity:

$$\{fg, h\} = f\{g, h\} + \{f, h\}g \quad \forall f, g, h \in C^\infty(M).$$

Equivalently, a Poisson structure can be given by a contravariant skew-symmetric 2-tensor P satisfying $[P, P] = 0$, where $[\ ,\ ]$ stands for the Schouten bracket. In local coordinates P can be written in the form:

$$P = \sum_{1\le i<j\le n} \{x_i, x_j\} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

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Weinstein's splitting theorem (see [7]) allows us to restrict to a neighbourhood of a zero rank point, if we are interested in the local structure of P.

To a Poisson tensor P one can associate a family of vector fields, the *curl vector fields of* P. One chooses a volume form Ω on the manifold M (which is assumed to be oriented), takes the exterior derivative of the $(n-2)$-form $i_P\Omega$, and then takes the pre-image of this $(n-1)$-form by the inverse map of the contraction with Ω. This vector field, $K_\Omega(P)$, is known as the *curl vector field of* P *with respect to* Ω. In local coordinates and taking $\Omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ we have

$$K_\Omega(P) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial P_{ij}}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

By changing the volume form, the curl vector field changes by sum with an Hamiltonian vector field (see [3]).

Such vector fields have been recently used in the local study of Poisson structures, mainly in the case of Poisson structures whose linear approximation at the singular point is trivial (e.g. quadratic Poisson structures). For example, the classification of quadratic Poisson structures in $\mathbf{R}^3$ has been done using an appropriate normal form for one of its curl vector fields (see [3]). As a corollary of our main result we refine such classification in the case the curl vector field is trivial, this being precisely the situation not dealt with in [3]. Some other important results have been obtained by assuming some kind of nondegeneracy condition on a curl vector field (hyperbolicity in [4] and invertibility in [5]).

We will consider Poisson structures P in $\mathbf{R}^3$ whose curl vector field (with respect to an appropriate volume form) is trivial (and therefore degenerate), and study them around a singular point. We will call such Poisson structures *closed* or *locally exact*. In other words, we assume that all curl vector fields of P are Hamiltonian. To each of these Poisson tensors we associate a smooth function. We will show that normal forms for these Poisson tensors can be obtained by putting the associated functions in normal form. We conclude by exhibiting normal forms (thus obtained) for some families of locally exact Poisson tensors.

2. Locally exact singular Poisson structures

2.1. Definitions and notation

We will assume from now on, without explicitly saying so, that all manifolds, Poisson tensors, functions and diffeomorphisms are either smooth or analytic. More important, all Poisson tensors will be assumed to be singular at some point. The normal forms obtained for such tensors are valid around the singular point.

Definition 1. Let (M, P) be an oriented Poisson manifold of dimension n. We say that P is *closed* (or *locally exact*) if there is a volume form Ω on M such that the $(n-2)$-form $i_P\Omega$ is closed. Equivalently, the curl vector field of P with respect to Ω, $K_\Omega(P)$, is trivial.

Remark 1. If P is locally exact and $\Omega' = a\Omega$ is another volume form on M, then $K_{\Omega'}(P) = X_{\ln|a|}$, the Hamiltonian vector field of the function $\ln|a|$. Conversely, if for some volume form Ω, the vector field $K_\Omega(P)$ is Hamiltonian, then P is locally exact.

2.2. The three-dimensional case

Now consider the case where P is a locally exact Poisson tensor on a manifold of dimension 3. Then the form $i_P\Omega$ is a closed 1-form. Since we are interested in the local structure of P around a singular point, we can assume that we are working in a neighbourhood of the origin in $\mathbf{R}^3$. Furthermore such 1-form is locally given by the differential of a function, so that there exists a function Ψ defined in a neighbourhood of the origin such that:

$$i_P\Omega|_U = \mathrm{d}\Psi.$$

We then say that P is *locally determined by* Ψ *(with respect to* Ω*)*, and write: $P = P_\Psi$. By subtracting $\Psi(0)$, if necessary, we can assume that $\Psi(0) = 0$.

Remark 2. The function Ψ is a local Casimir function for the Poisson tensor it determines.

In the next step we consider equivalence classes for functions in $\mathbf{R}^3$. Two functions Ψ and Ψ' (both vanishing at $x = 0$) are said to be *(locally) equivalent* if there exists a local diffeomorphism φ preserving the origin such that: $\Psi' = \Psi \circ \varphi$. In the following theorem we will translate the Poisson-equivalence of locally exact Poisson tensors in terms of the equivalence of functions which locally determine them.

Theorem 1. *If* $\Psi' = \Psi \circ \varphi$ *then* P_Ψ *(with respect to* Ω*) is Poisson-equivalent to* $P_{\Psi'}$ *(with respect to* $\phi^*\Omega$*).*

Proof. Let P denote the Poisson tensor P_Ψ with respect to a volume form Ω. Then

$$\mathrm{d}\Psi' = \varphi^*\,\mathrm{d}\Psi = \varphi^* i_P\Omega = i_{\varphi_*^{-1}P}(\varphi^*\Omega),$$

where $\varphi_*^{-1}P$ stands for the pushforward of P by φ^{-1}. This shows that the tensor $P' = \varphi_*^{-1}P$ is locally determined by $\mathrm{d}\Psi'$ with respect to $\Omega' = \varphi^*\Omega$. Since P' is, by definition, Poisson-equivalent to P, the proof is complete. □

Remark 3. The theorem implies that, whenever Ψ and Ψ' are equivalent functions, there exists a nowhere vanishing function ξ such that P_Ψ is Poisson-equivalent to $\xi P_{\Psi'}$ (now with respect to the *same* volume form). The problem of removing the function ξ from $\xi P_{\Psi'}$ is nontrivial, and often not possible.

3. Normal forms for locally exact Poisson structures in $\mathbf{R}^3$

In this section, we assume that the origin is not only singular but also a zero rank point for the locally exact Poisson structure P_Ψ. In such a case the function Ψ has a critical point at the origin.

3.1. The generic case

By the general singularity theory [2] we know that generic functions in $\mathbf{R}^3$ have only nondegenerate critical points and that their normal forms are:

1. $\Psi_1 = (x^2 + y^2 + z^2)/2$;
2. $\Psi_2 = (x^2 + y^2 - z^2)/2$;
3. $\Psi_3 = (x^2 - y^2 - z^2)/2$;
4. $\Psi_4 = -(x^2 + y^2 + z^2)/2$.

A corollary of Theorem 1 is the following theorem.

Theorem 2. *Generically, in the analytic category, locally exact singular Poisson structures in $\mathbf{R}^3$ are Poisson-equivalent to the Lie–Poisson tensor on either $so(3)^*$ or $sl(2, \mathbf{R})^*$.*

Proof. As remarked above, the normal forms for generic functions in $\mathbf{R}^3$ are Ψ_i, $i = 1, \dots, 4$. If we denote by P_i the Poisson tensor locally determined by Ψ_i (with respect to the volume form $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$), then an easy computation shows that P_1 is given by

$$P_1 = z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z},$$

which is just the Lie–Poisson tensor in the dual of $so(3)$. A linear change of coordinates shows that P_4 is Poisson-equivalent to P_1, so both (1) and (4) produce the same equivalence class for Poisson structures.

Also P_2 is the Lie–Poisson tensor on the dual of a semisimple Lie algebra. By looking at its symplectic leaves (which are precisely the level sets of the function Ψ_2), we can show that such Lie algebra is $sl(2, \mathbf{R})$. One can easily show that P_3 is Poisson-equivalent to P_2 so both (2) and (3) produce the Lie–Poisson structure on $sl(2, \mathbf{R})^*$.

This means that, generically, a locally exact singular Poisson tensor will be Poisson-equivalent to either ξP_1 or ξP_2 (see Remark 3). Now, in these two situations it is possible to get rid of the function ξ in the product ξP_i. To do this we just write $\xi = \xi_0 + \mathrm{O}(1)$ to produce:

$$\xi P_i = \xi_0 P_i + \mathrm{O}(2),$$

which is Poisson-equivalent (by a linear change of coordinates) to

$$P_i + \mathrm{O}(2).$$

We now use the result of Conn [1] to remove the perturbation $\mathrm{O}(2)$ from $P_i + \mathrm{O}(2)$. This is possible in the analytic category since both $so(3)$ and $sl(2, \mathbf{R})$ are semisimple Lie algebras. □

Remark 4. In the smooth situation the results of Conn can not be used to remove the perturbation $\mathrm{O}(2)$ from $P_2 + \mathrm{O}(2)$, since the Lie algebra $sl(2, \mathbf{R})$ is smoothly degenerate (see [7]).

3.2. The quadratic case

As another corollary of Theorem 1 we refine the linear classification given by Dufour and Haraki [3] for quadratic Poisson structures in $\mathbf{R}^3$. Equivalence class number 14 (as in that reference) is

$$\{x, y\} = \frac{\partial \Psi}{\partial z}, \qquad \{x, z\} = -\frac{\partial \Psi}{\partial y}, \qquad \{y, z\} = \frac{\partial \Psi}{\partial x},$$

where Ψ is an homogeneous polynomial of degree 3. We remark that all Poisson tensors in this class are locally exact Poisson tensors. Using a normal form for such polynomials and Theorem 1 we obtain the following result.

Theorem 3. *The (linear) equivalence classes for (singular) locally exact quadratic Poisson structures in* $\mathbf{R}^3$ *are*:

$$P = a\left(\frac{\partial \Psi}{\partial z}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} - \frac{\partial \Psi}{\partial y}\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + \frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}\right), \quad a \in \mathbf{R},$$

where Ψ *is one of the following*:

$$\Psi_1 = x^3, \qquad \Psi_2 = x^2 y, \qquad \Psi_3^{\pm} = x^3 \pm xz^2, \qquad \Psi_4 = x^2 z + yz^2,$$
$$\Psi_5 = x^3 + yz^2, \qquad \Psi_6 = xyz, \qquad \Psi_7 = x^3 + xyz, \qquad \Psi_8 = x^3 + y^3 + xyz,$$
$$\Psi_9 = x^3 + z(x^2 + y^2), \qquad \Psi_{10}^{\pm} = x^3 \pm x(y^2 + z^2),$$
$$\Psi_{11}^b = x^3 + y^3 + z^3 + bxyz \quad (b \in \mathbf{R}).$$

Remark 5. In the case Ψ is one of $\Psi_1, \ldots, \Psi_5$, the constant a can be removed from the normal form for P.

Proof. We refer to [6] for the above normal forms (by linear diffeomorphisms) for homogeneous polynomials of degree 3. Note that in the case of equivalence by linear diffeomorphisms, the function ξ in Remark 3 is constant. This implies that if Ψ is (linearly) equivalent to Ψ' then P_Ψ is Poisson-equivalent to $aP_{\Psi'}$ ($a \in \mathbf{R}$) with respect to the same volume form. We just have to make sure that the Poisson tensors locally determined by these functions (with respect to the volume form $\Omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$) are in different equivalence classes. The following lemma is a weak version of the converse of Theorem 1.

Lemma 1. *Suppose that* $P = P_\Psi$ and $P' = P_{\Psi'}$ (*with respect to the same volume form* Ω) *are Poisson-equivalent by a linear isomorphism* φ. *Then* Ψ *and* Ψ' *are equivalent by a linear isomorphism.*

Proof. By hypothesis we have

$$\mathrm{d}\Psi' = i_{P'}\Omega = i_{\varphi_*^{-1}P}\Omega = \varphi^* i_P (\varphi^{-1})^* \Omega.$$

Now, because φ^{-1} is a linear isomorphism we have

$$(\varphi^{-1})^* \Omega = \lambda \Omega$$

with $\lambda \neq 0$, which shows that:

$$\mathrm{d}\Psi' = \varphi^*(\lambda i_P \Omega) = \varphi^*(\mathrm{d}(\lambda \Psi)) = \mathrm{d}(\lambda \Psi \circ \varphi).$$

Finally, we remark that if Ψ and Ψ' are two homogeneous polynomials of degree 3 such that:

$$\Psi = \lambda \Psi' \circ \varphi$$

for some linear isomorphism φ and nonzero λ, then Ψ and Ψ' are in the same (linear) equivalence class. This completes the proof of the lemma. □

The lemma shows that different (linear) equivalence classes for homogeneous polynomials of degree 3 produce different (linear) equivalence classes for the Poisson tensors locally determined by them. The conclusion of the proof of Theorem 3 follows. □

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